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On the spectrum of AdS/CFT beyond supergravity

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Abstract: We test the spectrum of string theory on $AdS_5 \times S^5$ derived in [1] against that of single-trace gauge invariant operators in free $\mathcal{N} = 4$ super Yang-Mills theory. Masses of string excitations at *critical tension* are derived by extrapolating plane-wave frequencies at $g_{\rm YM} = 0$ down to finite J. On the SYM side, we present a systematic description of the spectrum of single-trace operators and its reduction to PSU(2, 2|4) superconformal primaries via a refined Eratostenes' supersieve. We perform the comparison of the resulting SYM/string spectra of charges and multiplicities order by order in the conformal dimension Δ up to $\Delta = 10$ and find perfect agreement. Interestingly, the SYM/string massive spectrum exhibits a hidden symmetry structure larger than expected, with bosonic subgroup SO(10, 2) and thirty-two supercharges.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.

Contents

1. Introduction

The strong form of Maldacena's conjecture [2] relates perturbative super Yang-Mills theory (SYM) to a higher spin (HS) gravity theory in its broken phase. At strictly zero gauge coupling $g_{\mathrm{YM}} = 0$, the HS symmetry is recovered. According to holography this suggests the existence of a "critical AdS radius" where infinitely many gauge particles come down to zero mass and the string spectrum on the highly curved $AdS_5 \times S^5$ should coincide with that of free SU(N) $\mathcal{N} = 4$ SYM theory. The dynamics at this point is practically frozen if not for the 'mixing' of single- with multi-particle states which is suppressed by $g_{\mathrm{s}} \approx 1/N$. In [1] the spectrum of Kaluza-Klein (KK) descendants of string excitations in $AdS_5 \times S^5$

was derived. The results were written in the deceivingly simple and suggestive form:

$$\mathcal{H}_{\text{AdS}} = \mathcal{H}_{\text{sugra}} + \mathcal{T}_{\text{KK}}\, \mathcal{T}_{\text{sconf}} \sum_{\ell=1}^{\infty} (\text{vac}_\ell \times \text{vac}_\ell)\,, \tag{1.1}$$

where vac_ℓ ($\ell = N_L = N_R$ denoting the string level) encodes the physical spectrum of chiral string primaries in flat space, properly rearranged in representations of SO(10) as we shall describe in this paper.[1] The supergravity states $\mathcal{H}_{\text{sugra}}$ organize into $\frac{1}{2}$-BPS multiplets, while massive string excitations ($\ell \geq 1$) sit in long multiplets given by tensoring string primaries with the long Konishi supermultiplet $\mathcal{T}_{\text{sconf}}$ [3, 4] on AdS. Finally, $\mathcal{T}_{\text{KK}} = \sum_n (n00; 00)$ is a polynomial of $\text{SO}(6) \times \text{SO}(4)$ representations accounting for Kaluza-Klein (KK) descendants.

Despite some progress [5]–[16], a viable quantization scheme for type IIB string theory on $AdS_5 \times S^5$ is still lacking. Linearized field equations around AdS are not available and therefore assignments of conformal dimensions Δ in (1.1) were missing. In [1], we exploited the higher spin symmetry enhancement in the *critical tension* limit[2] ($R^2 \approx \alpha'$) in order to fix the masses, i.e. dimensions, of states in the "first Regge trajectory"[3] $\Delta = 2 + s_{10} = 2\ell + n$ with string level ℓ and KK floor n. In particular, massless HS gauge particles correspond to states in the first Regge trajectory with $s_4 = 2\ell - 2$, and $n = 0$ [23]–[25]. Scalar fields dual to $\frac{1}{2}$-BPS operators and their superpartners are associated to KK recurrences of supergravity ($\ell = 0, n \geq 2$) states. Similarly, string primaries associated to marginal/relevant deformations of $\mathcal{N} = 4$ SYM were also found in the "first Regge trajectory" ($\ell \leq 2, n \leq 4$) and the right spectrum of charges and multiplicities was reproduced [1].

The purpose of this paper is to refine the above analysis and present a systematic description of the string/operator map at $g_{\text{YM}} = 0$ beyond the first Regge trajectory. We will assign conformal dimensions of states in (1.1) by setting first $g_{\text{YM}} = 0$ and then flowing to the plane-wave limit [26]. More precisely, exploiting the manifest SO(10) form of $\sum(\text{vac}_\ell \times \text{vac}_\ell)$ and choosing an $\text{SO}(2)_J \times \text{SO}(8)$ subgroup inside, we assign conformal dimensions Δ in such a way that:

$$\Delta \,-\, J = \nu\,, \tag{1.2}$$

with $\nu = \sum N_n w_n \;\rightarrow\; \sum_n N_n$ the plane-wave string frequencies at $g_{\text{YM}} = 0$. This is to be contrasted with the opposite limit, string theory in flat spacetime, where the hamiltonian measures the string level $\ell = \sum_n nN_n$. We will extrapolate formula (1.2) down to finite J along the line $g_{\text{YM}} = 0$. The perfect match between the string spectrum of conformal dimensions found in this way and that of SYM strongly suggests that this simple formula is indeed exact at $g_{\text{YM}} = 0$ and we apply it to the full string spectrum. This is a realization of the idea proposed in [27, 28] that the BMN spectrum can be extended to the generic

[1] The products in (1.1) are understood in terms of the $\text{SO}(6) \times \text{SO}(4)$ subgroup.

[2] The study of propagating strings on AdS near the HS critical radius has been recently addressed in [17]–[22].

[3] The quotation marks distinguish this from the more familiar Regge trajectory $\Delta = 2 + s_4$ with s_4 the four- (rather than the ten-)dimensional spin.

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finite J case and thus give the full spectrum a 'stringy' nature. The possibility of assembling states in irreducible 'massless' and 'massive' HS multiplets will only be hinted at in the conclusions and discussed in greater details in a companion paper [29].

On the SYM side, the counting of gauge invariant single-trace composite operators is performed in the framework of Polya theory [30] that allows to count 'words' of arbitrary length, i.e. 'letters' in a given alphabet, modulo the action of the cyclic group, for SU(N), in order to account for the cyclicity of the trace. This is tantamount to the more romantic problem of counting 'necklaces' made of an arbitrary number of 'beads' of given 'colors'.

In order to reduce 'entropy' in the comparison, we find it convenient to focus on 'superprimaries' of PSU$(2,2|4)$. We achieve this goal by refining the Eratostenes' Supersieve procedure introduced in [1] and further getting rid of KK recurrences so as to expose a hidden SO$(10,2)$ structure in the SYM spectrum beyond the $\frac{1}{2}$-BPS series $\mathcal{Z}_{\mathrm{BPS}}$: The AdS/CFT correspondence predicts that the spectrum of gauge invariant $\mathcal{N}=4$ SYM operators should take a form analogous to that of (1.1)

$$\mathcal{Z}_{\mathrm{SYM}} = \mathcal{Z}_{\mathrm{BPS}} + \mathcal{T}_{\mathrm{KK}}\ \mathcal{T}_{\mathrm{sconf}}\ \mathcal{Z}_{\mathrm{SO}(10,2)}\,. \tag{1.3}$$

In particular, this implies that SYM states beyond the $\frac{1}{2}$-BPS series will be organized in towers of superconformal and "Kaluza-Klein" descendants. Remarkably, these two towers can be combined into a single "multiplet" $\mathcal{T}_{\mathrm{SO}(10,2)} \equiv \mathcal{T}_{\mathrm{KK}}\ \mathcal{T}_{\mathrm{sconf}}$, which is generated by SO(10) supertranslations Q_{16}, P_{10} with $P_{10}^2 = 0$ (arising from the tracelessness condition for KK harmonics), suggesting a hidden SO$(10,2)$ structure, cf. (3.13) below. Indeed, acting with P_{10} lifts SO$(10)\times$SO(2) representations to SO$(10,2)$. This points towards the extension of the hidden 12-dimensional structures found in the KK towers of $AdS_5 \times S^5$ supergravity [31] to the entire massive string spectrum! On the SYM side this implies that all SYM states beyond the $\frac{1}{2}$-BPS series can be organized in representations of SO$(10,2)_\Delta$ rather than SO$(6)\times$SO$(4,2)_\Delta$. We have explicitly verified this structure till $\Delta = 10$.

Even stronger, the AdS/CFT correspondence yields a quantitative prediction obtained by comparing the spectra on the string (1.1) and SYM (1.3) side:

$$\mathcal{Z}_{\mathrm{SO}(10,2)} \stackrel{!}{\equiv} \sum_{\ell=1}^{\infty} (\mathrm{vac}_\ell \times \mathrm{vac}_\ell)\,. \tag{1.4}$$

As the main result of this paper, we have verified this relation as well up to $\Delta = 10$, supporting the conjectured mass formula (1.2).

At small but finite 't Hooft coupling, all but a handful of massless states become massive in a Pantagruelic Higgs mechanism, whereby higher spin multiplets in the bulk eat lower spin multiplets and become 'long' and 'massive'. The counterpart of this 'grande bouffe' in the boundary theory is the appearance of anomalous dimensions [4], [32]–[38] that, following [39], can be systematically and almost straightforwardly computed at one-loop for all gauge invariant operators thanks to the integrability of the associated 'super-spin chain' [40]–[42]. This is however beyond the scope of our analysis and will be taken care of (in a restricted yet interesting set of states) in [29]. The related problem of establishing integrability beyond one-loop [43] is the subject of active investigation [44]–[47] we have little to say about here.

The plan of the paper is as follows. In section 2, we review the naive procedure to compute the string spectrum on $AdS_5 \times S^5$ starting from the light-cone GS formalism in flat space. Exploiting the hidden SO(10) covariance and the BMN limit, we propose a very simple yet effective mass/dimension formula for all string states at the HS enhancement point dual to free SYM theory. In section 3, we compute the SYM spectrum by resorting to Polya theory, and describe how to identify HWS, i.e. superprimaries, by means of a refined Eratostenes' Supersieve. Removal of 'KK recurrences' further simplifies the problem of comparing the two spectra. We find perfect agreement for all the states we have explicitly analyzed, i.e. superprimaries with $\Delta \leq 10$ and their descendants. Section 4, contains our conclusions and perspectives. Various appendices tabulate our results and contain unwieldy but necessary formulae for the interested reader.

2. String states on $AdS_5 \times S^5$

In this section, we review the KK reduction of string theory on $AdS_5 \times S^5$ following [1], and propose a mass formula (2.17) for the entire string spectrum at critical tension, i.e. at the HS enhancement point dual to $g_{\rm YM} = 0$. To linear order in fluctuations around $AdS_5 \times S^5$ the type IIB field equations boil down to a set of uncoupled free massive equations

$$\left(\nabla^2_{AdS_5 \times S^5} - M^2_{\Phi}\right) \Phi_{\mathcal{R}} = 0\,, \tag{2.1}$$

with $\mathcal{R}$ labeling irreducible representations of $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$, the Lorentz group on $AdS_5 \times S^5$, and running over the spectrum of type IIB string excitations in flat space [1]. The form of eq. (2.1) is fixed by Lorentz covariance, while "masses" M^2_{Φ} describe the coupling of fields $\Phi_{\mathcal{R}}$ to the curvature and five-form flux. They can in principle be determined by explicit evaluation of the linearized equations around $AdS_5 \times S^5$ but these equations are not available beyond the supergravity level [23]–[25]. Even if this information is missing we will see how most of the information about the string spectrum can be derived from standard KK techniques while masses at *critical tension* can be fixed by requiring a consistent plane-wave limit.

The spectrum of KK harmonics is determined by group theory [48] (see [49]–[54] for applications in the AdS context). Expanding the ten-dimensional fields $\Phi_{\mathcal{R}}$ in S^5-spherical harmonics $\mathcal{Y}^r_{\mathcal{R}_{\mathrm{SO}(5)}}(y)$, leads to

$$\Phi_{\mathcal{R}}(x,y) = \sum_{r\,\in\,\mathrm{KK}[\mathcal{R}_{\mathrm{SO}(5)}]} \mathcal{X}^r_{\mathcal{R}_{\mathrm{SO}(1,4)}}(x)\,\mathcal{Y}^r_{\mathcal{R}_{\mathrm{SO}(5)}}(y)\,, \tag{2.2}$$

with $\mathcal{R} = \mathcal{R}_{\mathrm{SO}(1,4)} \times \mathcal{R}_{\mathrm{SO}(5)}$ and x, y being coordinates along AdS_5 and S^5 respectively. The sum runs over the set $\mathrm{KK}[\mathcal{R}_{\mathrm{SO}(5)}]$ of all SO(6) representations that contain $\mathcal{R}_{\mathrm{SO}(5)}$ in their decomposition under SO(5).

This program was carried out in [1]. The result for the massive string spectrum obtained this way may be written as

$$\mathcal{H}_{\rm AdS} = \mathcal{H}_{\rm sugra} + \mathcal{T}_{\rm KK}\,\mathcal{T}_{\rm sconf} \sum_{\ell=1}^{\infty} (\mathrm{vac}_\ell \times \mathrm{vac}_\ell)\,, \tag{2.3}$$

with $\mathcal{T}_{\rm KK}$, $\mathcal{T}_{\rm sconf}$ the KK and AdS superconformal descendant polynomials:[4]

$$\begin{aligned}
\mathcal{T}_{\rm KK} &= \sum_{n=0}^{\infty} (n00;00)^n \,, \\
\mathcal{T}_{\rm sconf} &= (1+Q+Q\wedge Q+\cdots)\,(1+P_4+(P_4\times P_4)_s+\cdots)\,, \\
Q &= \left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\right)^{1/2} , \qquad P_4=(000;10)^1\,,
\end{aligned} \tag{2.4}$$

of $\mathrm{SO}(6)\times\mathrm{SO}(4)$ representations. Here, $(n00;00)$ refers to the symmetric traceless representations of SO(6), singlet of SO(4), see appendix A for our notation. AdS supermultiplets $\mathcal{T}_{\rm sconf}$ are instead generated by 16 supersymmetries Q and $4d$ derivatives P_4 along the AdS_5 boundary. Finally $\sum_{\ell=1}^{\infty}(\mathrm{vac}_\ell\times\mathrm{vac}_\ell)$ denotes the massive string spectrum in flat space, rearranged in terms of $\mathrm{SO}(6)\times\mathrm{SO}(4)$ representations — this is achieved by lifting the original SO(9) representations up to SO(10), breaking down to $\mathrm{SO}(6)\times\mathrm{SO}(4)$ — and supplied with a mass quantum number Δ, denoted by a superscript $\mathcal{R}^\Delta$.

The spectrum (2.3) thus naturally assembles into long multiplets of PSU(2, 2|4)

$$\mathcal{A}^{\Delta}_{(w_1,w_2,w_3;w_4,w_5)} \equiv (w_1,w_2,w_3;w_4,w_5)^\Delta \times \mathcal{T}_{\rm sconf}\,. \tag{2.5}$$

At this point, the assignments of conformal dimensions are *ad hoc*; they will be justified later on in this section.

2.1 Strings on flat space, yet again

For completeness, we review here the spectrum of type II superstrings in flat space in the GS formulation. Chiral (say left-moving) string excitations are created by the raising modes α^I_{-n}, S^a_{-n} acting on the vacuum $|\mathcal{Q}_c\rangle$:

$$S^a_n|\mathcal{Q}_c\rangle = \alpha^I_n|\mathcal{Q}_c\rangle = 0\,, \qquad n>0\,. \tag{2.6}$$

Here and below, indices $I=1,\ldots,\mathbf{8_v}$, $a=1,\ldots,\mathbf{8_s}$, $\dot a=1,\ldots,\mathbf{8_c}$ run over the vector, spinor left and spinor right representations of the SO(8) little Lorentz group. In addition we have introduced the compact notations:

$$\mathcal{Q}_s \;=\; \mathbf{8_v}-\mathbf{8_s}\,, \qquad \mathcal{Q}_c \;=\; \mathbf{8_v}-\mathbf{8_c}\,, \tag{2.7}$$

to describe chiral worldsheet supermultiplets.[5] The vacuum $|\mathcal{Q}_c\rangle$ is 2^4-fold degenerate as a result of the quantization of the eight fermionic zero modes S^a_0. The physical spectrum of the type IIB superstring is defined by tensoring two (left and right moving) identical chiral spectra subject to the level matching condition $\ell\equiv N_L=N_R$. The spectrum T_ℓ at string excitation level ℓ takes the form

$$T_\ell = \mathcal{Q}_c^2 \times \left(\sum_{\sum r\,k_r=\ell}\prod_r \mathcal{Q}_s^{\cdot k_r}\right)^2 , \tag{2.8}$$

[4]The $\mathrm{SO}(4,2)$ content $D(\Delta|j_1,j_2)$ can be read from that of $\mathrm{SO}(4)\times\mathrm{SO}(2)$ presented here by omitting P_4 translations. This was the notation adopted in [1].

[5]The minus sign accounts for the spin/statistics.

where we use the notation of dotted (graded symmetrized) products according to

$$\begin{aligned}
\mathcal{Q}_s^{\cdot 2} &\equiv \mathbf{8}_{(v} \times \mathbf{8}_{v)} + \mathbf{8}_{[s} \times \mathbf{8}_{s]} - \mathbf{8}_v \times \mathbf{8}_s \;=\; \mathbf{8}_v \times \mathcal{Q}_s\,, \\
\mathcal{Q}_s^{\cdot 3} &\equiv \mathbf{8}_{(v} \times \mathbf{8}_v \times \mathbf{8}_{v)} - \mathbf{8}_{[s} \times \mathbf{8}_s \times \mathbf{8}_{s]} + \mathbf{8}_v \times \mathbf{8}_{[s} \times \mathbf{8}_{s]} - \mathbf{8}_s \times \mathbf{8}_{(v} \times \mathbf{8}_{v)} \\
&= (\mathbf{35} - \mathbf{8_c}) \times \mathcal{Q}_s\,. \\
&\text{etc.}
\end{aligned} \tag{2.9}$$

(In contrast, '$\times$' and '$\prod$' refer to the ordinary tensor product.) For the dotted products we further find the following recursive formula

$$\mathcal{Q}_s^{\cdot n} = \Xi_n \times \mathcal{Q}_s\,, \tag{2.10}$$

with

$$\Xi_n \;=\; \mathcal{L}(\Xi_{n-1}) + \begin{cases} -\mathbf{8_c} & n=3 \\ \mathbf{8_v} & n=4 \\ -\mathbf{8_c}+\mathbf{28} & n=5 \\ \mathbf{8_v}-\mathbf{8_s} & n>5 \text{ even} \\ \mathbf{1}-\mathbf{8_c}+\mathbf{28} & n>5 \text{ odd} \end{cases}\,, \qquad \mathcal{L}([k,l,p,q]) \equiv [k+1,l,p,q]\,.$$

For the first massive levels we then find explicitly

$$\begin{aligned}
T_1 &= \mathcal{Q}_c^2 \times \mathcal{Q}_s^2 = T_1 \times (\mathbf{1}_1)^2\,, \\
T_2 &= \mathcal{Q}_c^2 \times (\mathcal{Q}_s + \mathcal{Q}_s \cdot \mathcal{Q}_s)^2 = T_1 \times (\mathbf{1}_1 + \mathbf{8}_{\mathbf{v},2})^2\,, \\
T_3 &= \mathcal{Q}_c^2 \times (\mathcal{Q}_s + \mathcal{Q}_s^2 + \mathcal{Q}_s \cdot \mathcal{Q}_s \cdot \mathcal{Q}_s)^2 = T_1 \times (\mathbf{1}_1 + \mathbf{8}_{\mathbf{v},2} - \mathbf{8}_{\mathbf{s},2} + \mathbf{35}_{\mathbf{v},3} - \mathbf{8}_{\mathbf{c},3})^2\,;
\end{aligned} \tag{2.11}$$

for general $\ell > 0$, the structure is

$$T_\ell \equiv T_1 \times (\mathrm{vac}_\ell \times \mathrm{vac}_\ell) = T_1 \times (\textstyle\sum_j \mathcal{R}_{j,\nu})^2\,, \tag{2.12}$$

where the SO(8) content of the chiral ground states vac_ℓ is found by explicitly evaluating eqs. (2.8), (2.10). We have introduced the subscript "ν" for each SO(8) representation $\mathcal{R}_j$, indicating the excitation number, i.e. the number of $\mathcal{Q}_s = \{\alpha^I_{-n}, S^a_{-n}\}$ that generate the state $\mathcal{R}_{j,\nu}$ from the vacuum. Notice that disregarding this quantum number, the massive spectrum naturally assembles into SO(9) representations, as expected.

To proceed with the string spectrum on $AdS_5 \times S^5$, we note that the massive string spectrum in flat space may in fact be further lifted from SO(9) to SO(10) in terms of representations

$$[k,l,m,p,q]^* \equiv [k,l,m,p,q] - [k-1,l,m,p,q]\,, \qquad (k>0)\,. \tag{2.13}$$

For the first few string levels described by (2.11) one finds [6]

$$\begin{aligned}
\mathrm{vac}_1 &= [0,0,0,0,0]^1\,, \\
\mathrm{vac}_2 &= [1,0,0,0,0]^2 - [0,0,0,0,0]^3\,, \\
\mathrm{vac}_3 &= [2,0,0,0,0]^3 - [1,0,0,0,0]^4 + [0,0,0,0,1]^{5/2}\,,
\end{aligned} \tag{2.14}$$

[6]Bosonic (fermionic) states with negative (positive) multiplicities in vac_ℓ represent unphysical degrees of freedom. Note that after multiplying ($\mathrm{vac}_\ell \times \mathrm{vac}_\ell$) with the KK tower according to eq. (2.3), the unphysical states drop out [1].

with $[k,l,m,p,q]$ the Dynkin labels of SO(10), and the superscript Δ_L referring to the chiral contribution to the conformal dimensions. Here and in the following, $\Delta = \Delta_L + \Delta_R$ will always refer to the bare dimension, i.e. at $g_{\rm YM} = 0$, where the conformal dimensions (in general a complicated function of the AdS radius) reduce to integer or half-integer numbers. At this stage there is still an ambiguity in the lift to SO(10) due to the different spinor chiralities of SO(10). We shall comment on this in the next subsection.

The assignments for conformal dimensions Δ_L in eq. (2.14) have been chosen *ad hoc* such that the physical states saturate the SO(10) bound

$$\Delta_L[k,l,m,p,q] \geq 1 + k + 2l + 3m + \tfrac{5}{2}p + \tfrac{3}{2}q\,. \tag{2.15}$$

This bound emerges from the PSU(2,2|4) unitarity bound [55] after breaking SO(10) down to SO(6) $\times$ SO(4) $\subset$ PSU(2,2|4). The assignments (2.14) will be justified in the next subsection in a more general setting where string primaries with conformal dimensions beyond the unitarity bound will also be found at higher string levels.

Let us close this section by observing a peculiarity of the massive flat space string spectrum: the partial sums of chiral string excitations $\sum_{\ell=1}^{\ell_m} \mathrm{vac}_\ell$ naturally assemble into true SO(10) representations! E.g. $\mathrm{vac}_1 = \mathbf{1}$, $\mathrm{vac}_1 + \mathrm{vac}_2 = \mathbf{10}$, $\mathrm{vac}_1 + \mathrm{vac}_2 + \mathrm{vac}_3 = \mathbf{54} - \mathbf{16_s}$, and so on.

2.2 Mass formula

According to eq. (2.3), the string spectrum on $AdS_5 \times S^5$ may be derived from that of string theory in flat space, upon multiplying in the KK towers. To this end, the SO(10) content (2.14) of the chiral ground states vac_ℓ is supplied with an additional SO(2) quantum number Δ, the conformal dimension. Upon breaking SO(10) down to SO(6) $\times$ SO(4), they yield the highest weight states of the long multiplets (2.5) in which the string spectrum is organized. In this subsection, we will specify the conformal dimensions associated with the SO(10) content of string primaries. We focus on the HS symmetry enhancement (free SYM) point.

As discussed in the introduction, the free SYM limit is accessible from the plane-wave regime, where a mass formula describing conformal dimensions of operators with a large $\mathrm{SO}(2)_J \in \mathrm{SO}(6)$ charge J is available [26, 56, 57]

$$\Delta - J = \sum_n N_n w_n \equiv \sum_n N_n \sqrt{1 + \frac{g_{\rm YM}^2 N n^2}{J^2}}\,, \tag{2.16}$$

Following the conjecture of a stringy spectrum even outside the BMN regime [27, 28] we will argue that at $g_{\rm YM} = 0$, the masses for the entire string spectrum can be fixed by extrapolating this formula down to finite J.

Specifically, breaking the SO(10) spectrum (2.14) down to $\mathrm{SO}(8) \times \mathrm{SO}(2)_J$ yields states $[k,l,p,q]_J^\Delta$. These are to be identified with the string state $[k,l,p,q]_\nu$ with ν the string excitation number introduced in eq. (2.11) above. At $g_{\rm YM} = 0$ the BMN mass formula (2.16) then reads[7]

$$\Delta - J = \nu \equiv \sum_n N_n \, . \tag{2.17}$$

We hence propose this relation to hold on the entire massive string spectrum at $g_{\rm YM} = 0$. Given the SO(10) content of the flat space string spectrum eq. (2.12), equation (2.17) uniquely determines the bare dimensions Δ in the string spectrum on $AdS_5 \times S^5$. As an illustration, let us consider the first few string levels given above:

$$\begin{aligned}
\mathrm{vac}_1 \quad &= \quad [0,0,0,0,0]^1 \\
&\overset{\mathrm{SO}(8)\times\mathrm{SO}(2)}{\to} \quad [0,0,0,0]^1_0 \\
&\overset{\mathrm{eq.\ (2.17)}}{\to} \quad [0,0,0,0]_1 \, , \\
\mathrm{vac}_2 \quad &= \quad [1,0,0,0,0]^2 - [0,0,0,0,0]^3 \\
&\overset{\mathrm{SO}(8)\times\mathrm{SO}(2)}{\to} \quad [1,0,0,0]^2_0 + [0,0,0,0]^2_1 + [0,0,0,0]^2_{-1} - [0,0,0,0]^3_0 \\
&\overset{\mathrm{eq.\ (2.17)}}{\to} \quad [1,0,0,0]_2 + [0,0,0,0]_1 \, , \\
\mathrm{vac}_3 \quad &= \quad [2,0,0,0,0]^3 - [1,0,0,0,0]^4 - [0,0,0,0,1]^{5/2} \\
&\overset{\mathrm{SO}(8)\times\mathrm{SO}(2)}{\to} \quad [2,0,0,0]^3_0 + [1,0,0,0]^3_1 + [1,0,0,0]^3_{-1} + [0,0,0,0]^3_0 + [0,0,0,0]^3_2 + \\
&\qquad + [0,0,0,0]^3_{-2} - [1,0,0,0]^4_0 - [0,0,0,0]^4_1 - [0,0,0,0]^4_{-1} - \\
&\qquad - [0,0,0,1]^{5/2}_{1/2} - [0,0,1,0]^{5/2}_{-1/2} \\
&\overset{\mathrm{eq.\ (2.17)}}{\to} \quad [2,0,0,0]_3 + [1,0,0,0]_2 + [0,0,0,0]_1 - [0,0,1,0]_3 - [0,0,0,1]_2 \, ,
\end{aligned} \tag{2.18}$$

which precisely reproduces eq. (2.11) and thus confirms our *ad hoc* assignments (2.14)! For these low massive levels, the conformal dimensions determined by eq. (2.17) all saturate the unitarity bound (2.15). At higher levels, starting from the $\Delta = 3$ singlet at level $\ell = 5$, this bound is still satisfied but no longer saturated; the correct conformal dimensions are rather obtained from eq. (2.17).

To summarize, the massive flat space string spectrum may be lifted to $\mathrm{SO}(10)\times\mathrm{SO}(2)_\Delta$, such that breaking SO(10) down to $\mathrm{SO}(8) \times \mathrm{SO}(2)_J$ reproduces the original SO(8) string spectrum and its excitation numbers eq. (2.11) via the relation (2.17). The results up to string level $\ell = 10$ are displayed in appendix B. For these levels, relation (2.17) not only determines the conformal dimensions Δ, but also fixes all ambiguities, arising in the lift of the SO(9) massive string spectrum to SO(10). Here, superconformal and KK descendants are included by replacing the flat long multiplet T_1 by the product of the long Konishi multiplet $\mathcal{T}_{\rm sconf}$ and the tower of KK descendants $\mathcal{T}_{\rm KK}$

$$T_1 \to \mathcal{T}_{\rm KK}\, \mathcal{T}_{\rm sconf} \, . \tag{2.19}$$

Eventually, breaking this SO(10) down to $\mathrm{SO}(6) \times \mathrm{SO}(4)$ then yields the massive string spectrum on $AdS_5 \times S^5$ via eq. (2.3). Based on the relation (2.17), we have thus been able

[7]This limit is to be contrasted with that of string theory in flat space, where the string hamiltonian measures the occupation number $\ell = \sum_n nN_n$ rather than $\nu = \sum_n N_n$. Note that the level matching condition $\ell = \bar{\ell}$ in contrast is left unaltered by sending $g_{\rm YM}$ to zero.

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to assign the string masses at $g_{\rm YM} = 0$. It would clearly be interesting, to achieve a direct derivation of this result. In appendix C, we have formulated some of these results in terms of the string partition function. In the next section we shall compare the string spectrum to $\mathcal{N} = 4$ SYM theory.

3. $\mathcal{N} = 4$ SYM theory

The spectrum of KK descendants of fundamental string states can be precisely tested against that of gauge invariant operators in SU(N) $\mathcal{N} = 4$ SYM theory. In this subsection we apply Polya theory [30, 58, 59], (see [1] for a quick review) and the Super-sieve algorithm introduced in [1] to determine the spectrum of superconformal primaries in $\mathcal{N} = 4$ SYM theory. To facilitate the reading of this section, we first explain the algorithm in full generality and display in section 3.3 explicit expressions only for the 'blind partition function', that counts states according to their conformal dimensions, independently of the remaining quantum numbers. The full spectrum of multiplicities and quantum numbers will be displayed in the appendix.

3.1 The single trace spectrum

The spectrum of gauge invariant $\mathcal{N} = 4$ SYM operators is organized under the supergroup PSU(2, 2|4). Multiplicities will be encoded in the weighted partition function:

$$\mathcal{Z}_n(t, y_i) = \mathrm{Tr}_{n-\mathrm{letters}}(-)^F\, t^{\Delta}\, \mathbf{y}^{\mathbf{w}}\,, \qquad \mathbf{y}^{\mathbf{w}} \equiv \prod_{i=1}^{5} y_i^{w_i}\,, \tag{3.1}$$

where the vector $\mathbf{w} = (w_1, w_2, w_3; w_4, w_5)$ and Δ label the charges under the canonical $\mathrm{SO}(2)^3\times\mathrm{SO}(2)^2\times\mathrm{SO}(2)_\Delta$ Cartan generators of the $\mathrm{SO}(6)\times\mathrm{SO}(4)\times\mathrm{SO}(2)_\Delta$ compact bosonic subgroup of PSU(2, 2|4). The insertion $(-)^F$, F being the fermion number, accounts for the right spin statistics. We restrict ourselves to single trace operators, i.e. for SU(N), cyclic words built from SYM letters: ϕ^i, λ^A_α, $\lambda_{A\dot\alpha}$, $F_{\mu\nu}$ and derivatives thereof.[8]

A word consisting of a single letter is clearly cyclic. The field content of single-letter words is then encoded in

$$\begin{aligned}\mathcal{Z}_1(t, y_i) &= \sum_{s=0}^{\infty}\left[t^{s+1}\,\partial^s\phi + t^{s+3/2}\,\partial^s\lambda + t^{s+3/2}\,\partial^s\bar\lambda + t^{s+2}\,\partial^s F\right]\\ &= \sum_{s=0}^{\infty}\Big[\chi_{(100)}\chi_{(s,0)}\,t^{s+1} + \chi_{(000)}\chi_{(s+1,1)}\,t^{s+2} + \chi_{(000)}\chi_{(s+1,-1)}\,t^{s+2} -\\ &\qquad - \chi_{(1/2,1/2,1/2)}\chi_{(s+1/2,1/2)}\,t^{s+3/2} - \chi_{(1/2,1/2,-1/2)}\chi_{(s+1/2,-1/2)}\,t^{s+3/2}\Big]\,,\end{aligned} \tag{3.2}$$

where $\chi_{(w_1,w_2,w_3)}\chi_{(w_4,w_5)}$ denotes the character polynomials of SO(6)×SO(4) representation with highest weight state $(w_1, w_2, w_3; w_4, w_5)$. They are given by formulas (A.3),(A.4) in

[8] The indices $i = 1, \ldots, 6$, $\mu = 0, \ldots, 3$, label the vector representations of SO(6) and SO(4), while $A = 1, \ldots, 4$, and $\alpha, \dot\alpha = 1, 2$ label the spinor representations.

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appendix A. Cyclic words with $n > 1$ letters are given in terms of Polya's formula [1]

$$\mathcal{Z}_{\rm SYM}(t, y_i) = \sum_{n=2}^{\infty} \mathcal{Z}_n(t, y_i) = \sum_{n, n|d} \frac{\varphi(d)}{n} \, \mathcal{Z}_1(t^d, y^d)^{n/d} \,, \tag{3.3}$$

with the sum running over all integers n and their divisors d, and Euler's totient function $\varphi(d)$ equals the number of integers relatively prime to d and smaller than d with $\varphi(1) = 1$ by definition. The omission of the $n = 1$ term in the sum is due to the fact that we are considering SYM with gauge group SU(N) rather than U(N).

The set of states in eq. (3.3) organizes into multiplets of the $\mathcal{N} = 4$ superconformal algebra PSU$(2, 2|4)$. The spectrum of superconformal primaries can be found by filtering eq. (3.3) by a sort of Eratostenes' Sieve, that removes at each step all descendants from superconformal primaries and declare "primaries" the remaining lowest conformal dimension states [1].

As the first step, we identify superconformal primaries in $\mathcal{Z}_{\rm SYM}(t, y_i)$. According to eq. (2.5), the character polynomial $\chi_{\mathbf{w}}^{\Delta}$ of a generic long supermultiplet of PSU$(2, 2|4)$ with highest weight state $t^{\Delta}\, \mathbf{y}^{\mathbf{w}}$ is generated by all (super)translations in the following way (Racah-Speiser algorithm)

$$\chi_{\mathbf{w}}^{\Delta}(t, y_i) = \mathcal{T}_{\rm sconf}(t, y_i)\, t^{\Delta}\, \chi_{\mathbf{w}}(y_i) \,, \tag{3.4}$$

where

$$\mathcal{T}_{\rm sconf}(t, y_i) = \frac{\prod_{s=1}^{16} \left(1 - t^{1/2} \mathbf{y}^{\mathbf{w}_s}\right)}{\prod_{v=1}^{4} (1 - t\, \mathbf{y}^{\mathbf{w}_v})} \,, \tag{3.5}$$

denotes the character polynomial of the long supermultiplet $\mathcal{T}_{\rm sconf}$ from eq. (2.4). The vectors $\mathbf{w}_s, \mathbf{w}_v$ in eq. (3.5) run over the weights of the 16-spinor and the 4-vector representation of the supersymmetry Q and translation generator P, respectively,

$$\begin{aligned} \mathbf{w}_{s=1\ldots16} &= \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}; \pm\frac{1}{2}, \pm\frac{1}{2}\right) \,, \qquad \text{with} \prod_{i=1}^{5} w_{s,i} > 0 \,, \\ \mathbf{w}_{v=1\ldots4} &= (0, 0, 0; \pm 1, 0), (0, 0, 0; 0, \pm 1) \,. \end{aligned} \tag{3.6}$$

When interactions are turned on (but still at large N to avoid mixing with multi-trace operators for which further shortening conditions may apply), single-trace operators fall into two classes of PSU$(2, 2|4)$ multiplets: $\frac{1}{2}$-BPS and long multiplets [1]:

$$\mathcal{Z}_{\rm SYM}(t, y_i) = \mathcal{Z}_{\rm BPS}(t, y_i) + \mathcal{Z}_{\rm long}(t, y_i) \,, \tag{3.7}$$

where

$$\mathcal{Z}_{\rm BPS}(t, y_i) = \sum_{n=2}^{\infty} \chi_{(n00;00)}^{n}(t, y_i) \,, \tag{3.8}$$

is the partition function counting $\frac{1}{2}$-BPS states and their (super)descendants which corresponds to the supergravity spectrum. The remaining $\mathcal{Z}_{\rm long}(t, y_i)$ is the main subject of our investigations. States in $\mathcal{Z}_{\rm long}(t, y_i)$ sit in long multiplets of the superconformal algebra

PSU(2, 2|4). This matches the multiplet structure of the string result (2.3) and therefore comparisons to string theory can be restricted to superconformal primaries. Superconformal primaries can be found by factoring out $\mathcal{T}_{\rm sconf}$ in eq. (3.7):

$$\mathcal{Z}_{\rm sconf}(t, y_i) = \frac{\mathcal{Z}_{\rm long}(t, y_i)}{\mathcal{T}_{\rm sconf}(t, y_i)} . \tag{3.9}$$

3.2 Comparison to the string spectrum

The AdS/CFT correspondence predicts that the spectrum of superconformal primaries in SU(N) $\mathcal{N} = 4$ SYM matches that of string theory on $AdS_5 \times S^5$:

$$\mathcal{Z}_{\rm sconf} = \mathcal{T}_{\rm KK} \sum_{\ell=1}^{\infty} (\mathrm{vac}_\ell \times \mathrm{vac}_\ell) , \tag{3.10}$$

with

$$\begin{aligned} \mathcal{T}_{\rm KK} &\equiv \sum_{n=0}^{\infty} t^n \chi_{(n00)} = (1 - t^2) \prod_{v=1}^{6} (1 - t\, \mathbf{y}^{\mathbf{q}_v})^{-1} , \\ \mathbf{q}_{v=1\ldots6} &= (\pm1, 0, 0; 00), (0, \pm1, 0; 00), (0, 0, \pm1; 00) . \end{aligned} \tag{3.11}$$

The comparison is considerably simplified if one factorizes the contribution of "KK descendants" from both sides in eq. (3.10). We denote by $\mathcal{Z}_{\rm SO(10,2)}$ the partition function of SYM superconformal primaries up to KK recurrences and $\frac{1}{2}$-BPS states. According to eq. (3.9) one finds:

$$\mathcal{Z}_{\rm SO(10,2)} \equiv \frac{\mathcal{Z}_{\rm sconf}}{\mathcal{T}_{\rm KK}} \equiv \frac{\mathcal{Z}_{\rm long}}{\mathcal{T}_{\rm SO(10,2)}} . \tag{3.12}$$

Remarkably, KK and superconformal descendants can be combined together into a manifestly SO(10) × SO(2) covariant "supermultiplet":

$$\mathcal{T}_{\rm SO(10,2)} = \mathcal{T}_{\rm KK} \mathcal{T}_{\rm sconf} = (1 - t^2) \frac{\prod_{s=1}^{16} (1 - t^{\frac{1}{2}} \mathbf{y}^{\mathbf{w}_s})}{\prod_{v=1}^{10} (1 - t\, \mathbf{y}^{\mathbf{w}_v})} . \tag{3.13}$$

The appearance of $\mathcal{T}_{\rm SO(10,2)}$ that naturally extends the SCA to account for KK descendants is rather suggestive. Translations P_{10} in $\mathcal{T}_{\rm SO(10,2)}$ now carry the weights of a ten-dimensional vector in contrast with its four dimensional cousins in eq. (3.4). The factor $(1-t^2)$ subtracts the trace of KK recurrences and lifts to $P_{10}^2 = 0$. $\mathcal{T}_{\rm SO(10,2)}$ can be thought as the defining "Konishi-like" multiplet of a larger superalgebra with SO(10, 2) bosonic generators and thirty-two supercharges. A candidate for this algebra will be briefly described at the end of this section. Notice that since $\mathcal{T}_{\rm SO(10,2)}$ is manifestly covariant under SO(10, 2), the full massive SYM spectrum can be rearranged into representations of SO(10, 2) rather than SO(6) × SO(4, 2) if $\mathcal{Z}_{\rm SO(10,2)}(t, y_i)$ does. We have explicitly checked that this is the case for all SYM primaries with $\Delta \leq 10$.

The AdS/CFT prediction can now be stated in the simple form

$$\mathcal{Z}_{\rm SO(10,2)} \stackrel{!}{=} \sum_{\ell=1}^{\infty} (\mathrm{vac}_\ell \times \mathrm{vac}_\ell) . \tag{3.14}$$

This can be verified order by order in Δ. In particular, to order $\Delta_{\rm max}$ only terms in eq. (3.3) with $n < \Delta_{\rm max}$ letters contribute. We have carried out this program till $\Delta_{\rm max} = 10$. The spectrum of superconformal primaries is displayed in appendix D (however due to the limited space only until $\Delta = \frac{13}{2}$). Comparison with the string theory results for vac$_\ell$ with $\ell \leq 10$ collected in appendix B show perfect agreement till $\Delta_{\rm max} = 10$! These results strongly support the conjectured dimension formula (2.17).

Before leaving this section we would like to comment on a candidate realization for the superalgebra underlying the SYM/string massive spectrum. The natural superalgebra is $Osp(1|32)$, i.e. the superalgebra generated by 32 'real' supercharges. The most familiar representation of this algebra is probably the M-theory superalgebra in $D = 11$, with signature (10,1). A less familiar representation (the relevant to us) is what might deserve the name of F-theory superalgebra in $D = 12$, with signature (10,2) where 32-dimensional Majorana-Weyl spinors exist, it reads [60]

$$\{\mathcal{Q}_a, \mathcal{Q}_b\} = J_{MN}\Gamma^{MN}_{ab} + Z^{+}_{MNPQRS}\Gamma^{MNPQRS}_{ab}\,, \tag{3.15}$$

where Γ's are $\mathrm{SO}(10,2)$ Dirac matrices and $\Gamma^{13} = +1$. The J_{MN}, Z^{+}_{MNPQRS} are some sort of 'generalized' Lorentz generators and central charges respectively. The absence of momenta and the presence of two times suggests the impossibility of realizing this superalgebra in a (super)Poincarè invariant vacuum. It is also impossible to reduce it to $D = 10$ with signature (9,1) and get the centrally extended type IIB superlagebra. Yet this superalgebra contains the isometry group of $AdS_5 \times S^5$ and matches the supersymmetries in the AdS superalgebra $\mathrm{PSU}(2,2|4)$.[9] Indeed setting the central charges $J_{m,i}$, and all Z^{+}_{MNPQRS} to zero one gets

$$\begin{aligned} \{\bar{\mathcal{Q}}^A_\alpha, \mathcal{Q}^\beta_B\} &= \frac{1}{2}J_{mn}(\Gamma^{mn})_\alpha{}^\beta\delta^A{}_B + \frac{1}{2}T_{ij}(\Gamma^{ij})^A{}_B\delta_\alpha{}^\beta \\ \{\mathcal{Q}^\alpha_A, \mathcal{Q}^\beta_B\} &= 0 \qquad \{\bar{\mathcal{Q}}^A_\alpha, \bar{\mathcal{Q}}^B_\beta\} = 0\,. \end{aligned} \tag{3.16}$$

Here $m,n = 0,0',1,\ldots,4$, $i,j = 1,\ldots 6$ are $\mathrm{SO}(4,2)$ and $\mathrm{SO}(6)$ vector indices respectively while $\alpha,\beta = 1,\ldots,4$, $A,B = 1,\ldots,4$ run over the corresponding spinor representation. The $\mathrm{SO}(4,2)\times\mathrm{SO}(6) \in \mathrm{PSU}(2,2|4)$ bosonic subgroup is realized by J_{mn}, T_{ij}. The physical implications of this "symmetry structure" remain to be explored.

3.3 SYM partition function

In this subsection, we illustrate our algorithm by focusing on the 'blind partition function' $\mathcal{Z}_{\rm SYM}(t) \equiv \mathcal{Z}_{\rm SYM}(t,1)$. The full spectrum of SYM charges and multiplicities is obtained in a similar way and the results are displayed in appendix D. The starting point is the one-letter partition function (3.2) at $y_i = 1$:

$$\mathcal{Z}_1(t) = \frac{2t(3+\sqrt{t})}{(1+\sqrt{t})^3}\,. \tag{3.17}$$

[9] The results in [61] seems to indicate that $\mathrm{SU}(2,2|4)$ is not a subalgebra of $Osp(1|32)$.

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Plugging this in (3.3) one finds

$$\begin{aligned}
\mathcal{Z}_{\mathrm{SYM}}(t) &= \sum_{n=2}^{\infty} \sum_{n|d} \frac{\varphi(d)}{n} \left[\frac{2t^d(3+t^{\frac{d}{2}})}{(1+t^{\frac{d}{2}})^3} \right]^{n/d} && (3.18) \\
&= 21\,t^2 - 96\,t^{5/2} + 376\,t^3 - 1344\,t^{7/2} + 4605\,t^4 - 15456\,t^{9/2} + 52152\,t^5 - \\
&\quad - 177600\,t^{11/2} + 608365\,t^6 - 2095584\,t^{13/2} + 7262256\,t^7 - 25299744\,t^{15/2} + \\
&\quad + 88521741\,t^8 - 310927104\,t^{17/2} + 1095923200\,t^9 - 3874803840\,t^{19/2} + \\
&\quad + 13737944493\,t^{10} + \mathcal{O}(t^{21/2})\,. && (3.19)
\end{aligned}$$

The next step in our program is to identify superconformal primaries in $\mathcal{Z}_{\mathrm{SYM}}(t)$. This can be easily done by first subtracting from (3.18) states in the 1/2 BPS series (3.8)

$$\begin{aligned}
\mathcal{Z}_{\mathrm{BPS}}(t) &= \sum_{n=2}^{\infty} \frac{t^n\left(n+2-(n-2)t^{1/2}\right)}{12(1+t^{\frac{1}{2}})^4} \Big[(n+1)(n+3)\Big((n+2) - 3t^{1/2}(n-2)\Big) + \\
&\qquad + t(n-1)(n-3)\Big(3(n+2) - t^{1/2}(n-2)\Big)\Big] \\
&= \frac{t^2\left(20 + 80t^{1/2} + 146t + 144t^{3/2} + 81t^2 + 24t^{5/2} + 3t^3\right)}{(1-t)(1+t^{\frac{1}{2}})^8}\,, && (3.20)
\end{aligned}$$

and then the KK and supersymmetry descendants, i.e. dividing by

$$\mathcal{T}_{\mathrm{SO}(10,2)} = (1-t^2)\frac{(1-t^1/2)^{16}}{(1-t)^{10}}\,. \qquad (3.21)$$

One finds

$$\begin{aligned}
\mathcal{Z}_{\mathrm{SO}(10,2)}(t) &= \frac{[\mathcal{Z}_{\mathrm{SYM}}(t) - \mathcal{Z}_{\mathrm{BPS}}(t)]}{\mathcal{T}_{\mathrm{SO}(10,2)}(t)} && (3.22) \\
&= t^2 + 100\,t^4 + 236\,t^5 - 1728\,t^{11/2} + 4943\,t^6 - 12928\,t^{13/2} + \\
&\quad + 60428\,t^7 - 201792\,t^{15/2} + 707426\,t^8 - 2550208\,t^{17/2} + \\
&\quad + 9101288\,t^9 - 32568832\,t^{19/2} + 116831861\,t^{10} + \mathcal{O}(t^{21/2})\,. && (3.23)
\end{aligned}$$

The expansion (3.22) can be rewritten as

$$\begin{aligned}
\mathcal{Z}_{\mathrm{SO}(10,2)}(t) &= t^2 + (10t^2 - t^3)^2 + (-16t^{5/2} + 54t^3 - 10t^4)^2 + \\
&\quad + (45t^3 - 144t^{\frac{7}{2}} + 210t^4 + 16t^{\frac{9}{2}} - 54t^5)^2 + \cdots\,, && (3.24)
\end{aligned}$$

etc., in perfect agreement with (3.14) and the string spectrum (2.14), appendix B.

4. Discussion

We have analyzed the spectrum of string theory on $AdS_5 \times S^5$ at the HS enhancement point, exploiting the mass formula (2.17), and compared to that of single-trace gauge invariant operators in free SU(N) $\mathcal{N} = 4$ SYM theory. Up to $\Delta_{\max} = 10$ we found that the spectrum indeed organizes into SO(10, 2) representations, and confirms the AdS/CFT prediction (1.3). There are additional SYM quantum numbers at $g_{\mathrm{YM}} = 0$ whose string origin is not completely clear.

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Classical type IIB supergravity is invariant under the $\mathrm{U}(1)_B$ compact 'isotropy' subgroup of $\mathrm{SL}(2,\mathbb{R})$ that acts by chiral transformations on the fermions. This symmetry is anomalous at one-loop and is broken by string interactions. However, as originally observed by Intriligator [62, 63], the structure of the type IIB effective action suggests that amplitudes with up to four external supergravity states should be $\mathrm{U}(1)_B$ invariant even after including higher derivative terms that receive non-perturbative corrections [64]–[67]. In order for this 'bonus symmetry' to be present at tree level, the exchange of massive string states should not spoil it. This in turn implies that consistent assignments of $\mathrm{U}(1)_B$ charges to massive string states should be feasible in principle, e.g. by identifying allowed decay channels into two supergravity states with known $\mathrm{U}(1)_B$ charges. Three-point amplitudes with all massive states would however violate $\mathrm{U}(1)_B$. From the holographically dual viewpoint of $\mathcal{N}=4$ SYM, $\mathrm{U}(1)_B$ is an external automorphism of the superconformal algebra [68]. It acts as a chiral transformation of the four gaugini combined with a duality rotation of the field-strengths and as such it cannot possibly be a symmetry of the theory at $g_{\mathrm{YM}} \neq 0$ if not for a restricted class of correlation functions, i.e. any 2-point functions, 3-point functions with at most one operator in a long multiplet and 4-point functions of operators in 1/2-BPS multiplets.

The story of another quantum number, the length L of a single-trace operator is even more obscure. In SYM perturbation theory it is a perfectly good quantum number up to and including one-loop. The (generalized) Konishi anomaly and other anomalous effects [69, 47, 70] imply the mixing of operators with different lengths, i.e. different number of constituents, even at large N. Instanton effects, which are however highly suppressed at large N, wash out any memory of this quantum number not differently from larger orders in perturbation theory so it is very hard to say how one could even in principle assign this quantum number to string states before the HS enhancement point is reached.[10]

Other discrete quantum numbers allow for a more direct string interpretation. The parity P identified in [43] in the SYM spectrum is nothing but worldsheet parity Ω that survives as a symmetry after compactification on S^5 and is 'gauged' when S^5 is replaced by RP^5 with or without discrete two-form fluxes [71]–[74]. The holographic counterpart of the orientifold projection is the breaking of the $\mathrm{SU}(N)$ gauge group to $\mathrm{SO}(2N)$, $\mathrm{SO}(2N+1)$ or $Sp(2N)$, depending on the choice of two-form flux [71, 72]. We have confirmed the correspondence of P and Ω by including parity quantum numbers in eq. (3.14). In string theory, parity is obtained by the formula

$$(\mathrm{vac}_\ell \times \mathrm{vac}_\ell) = (\mathrm{vac}_\ell \times \mathrm{vac}_\ell)_+^{\Omega=+} + (\mathrm{vac}_\ell \times \mathrm{vac}_\ell)_-^{\Omega=-} . \tag{4.1}$$

This means the (anti)symmetric components in this tensor product have positive (negative) parity.[11] This implies in particular, that all fermions in $(\mathrm{vac}_\ell \times \mathrm{vac}_\ell)$, which need to be products of one fermion and one boson, always come in both parities $\Omega = \pm$.

[10]Nevertheless, B and L seem to be related in some sense and possibly the combinations $L \pm B$ can be associated to the left- and right-moving sectors of string theory, respectively.

[11]Note that symmetrization is understood in a graded sense, i.e. products two fermions receive the opposite symmetry.

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A. Notation of representations

In this appendix, we collect our notations for representations of the various groups appearing in the main text. In general, we denote representations $\mathcal{R}$ by their highest weight states which are specified by their weights $(w_1, \dots w_n)$ or their Dynkin labels $[a_1, \dots, a_n]$. For the groups SO(4), SO(6), and SO(10), used in the text, the change to the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, SU(4) and SO(10) Dynkin basis is simply given as

$$
\begin{aligned}
(s_1, s_2) &= [s_1 + s_2, s_1 - s_2]\,, \\
(j_1, j_2, j_3) &= [j_2 + j_3, j_1 - j_2, j_2 - j_3]\,, \\
(w_1, w_2, w_3, w_4, w_5) &= [w_1 - w_2, w_2 - w_3, w_3 - w_4, w_4 + w_5, w_4 - w_5]\,, \\
[s_1, s_2] &= \left(\tfrac{1}{2}s_1 + \frac{1}{2}s_2, \frac{1}{2}s_1 - \frac{1}{2}s_2\right), \\
[q_1, p, q_2] &= \left(p + \frac{1}{2}q_1 + \frac{1}{2}q_2, \frac{1}{2}q_1 + \frac{1}{2}q_2, \frac{1}{2}q_1 - \frac{1}{2}q_2\right), \\
[k, l, m, r_1, r_2] &= \left(k + l + m + \frac{1}{2}r_1 + \frac{1}{2}r_2, l + m + \frac{1}{2}r_1 + \frac{1}{2}r_2, m + \frac{1}{2}r_1 + \frac{1}{2}r_2,\right. \\
&\qquad \left.\frac{1}{2}r_1 + \frac{1}{2}r_2, \frac{1}{2}r_1 - \frac{1}{2}r_2\right), \qquad \text{(A.1)}
\end{aligned}
$$

respectively .

By $\chi_{\mathbf{w}}(y_i)$, we denote the character polynomial associated to the highest weight representation $\mathcal{R}_{\mathbf{w}}$ of SO(6) or SO(4):

$$
\chi_{\mathbf{w}}(y_i) \equiv \sum_{\mathbf{w}' \in \mathcal{R}_{\mathbf{w}}} \mathbf{y}^{\mathbf{w}'}\,, \qquad \mathbf{y}^{\mathbf{w}'} \equiv \prod_i y_i^{w'_i}\,. \qquad \text{(A.2)}
$$

The set of $\mathbf{w}'$ over which the sum runs may be determined recursively, using Freudenthal's multiplicity formula, or directly by use of Weyl's character formula, cf. appendix E.

E.g., for the first few irreps ($\mathbf{1}$, $\mathbf{6}$, $\mathbf{4}$, and $\mathbf{4}^*$) of SO(6) one has

$$
\begin{aligned}
\chi_{(000)} &= 1\,, \\
\chi_{(100)} &= y_1 + y_1^{-1} + y_2 + y_2^{-1} + y_3 + y_3^{-1}\,, \\
\chi_{(1/2,1/2,1/2)} &= y_1^{1/2}y_2^{1/2}y_3^{1/2} + y_1^{1/2}y_2^{-1/2}y_3^{-1/2} + y_1^{-1/2}y_2^{-1/2}y_3^{1/2} + y_1^{-1/2}y_2^{1/2}y_3^{-1/2}\,, \\
\chi_{(1/2,1/2,-1/2)} &= y_1^{1/2}y_2^{1/2}y_3^{-1/2} + y_1^{1/2}y_2^{-1/2}y_3^{1/2} + y_1^{-1/2}y_2^{1/2}y_3^{1/2} + y_1^{-1/2}y_2^{-1/2}y_3^{-1/2} \qquad \text{(A.3)}
\end{aligned}
$$

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while a generic irrep of SO(4) is given by

$$\chi_{(s_1,s_2)} = \frac{y_4^{-s_2}y_5^{-s_1} + y_4^{s_2}y_5^{2+s_1} - y_4^{s_1+1}y_5^{s_2+1} - y_4^{-1-s_1}y_5^{1-s_2}}{(1-y_4y_5)(1-y_4^{-1}y_5)} . \tag{A.4}$$

Tensor products of representations $\mathcal{R}, \mathcal{R}'$ translate into ordinary product of their character polynomials: $\chi_{\mathcal{R}\times\mathcal{R}'} = \chi_{\mathcal{R}}\chi_{\mathcal{R}'}$.

B. String primaries

In this appendix, we give a list of the first ten massive flat space string levels, organized under $SO(10) \times SO(2)_\Delta$, as described in section 2. We use the notation

$$[k,l,m,p,q]^* \equiv [k,l,m,p,q] - [k-1,l,m,p,q] , \qquad (k>0) . \tag{B.1}$$

$\ell = 1$:

Δ	$\mathcal{R}$
1	$[0,0,0,0,0]$

(B.2)

$\ell = 2$:

Δ	$\mathcal{R}$
2	$[1,0,0,0,0]^*$

(B.3)

$\ell = 3$:

Δ	$\mathcal{R}$
3	$[2,0,0,0,0]^*$
$\frac{5}{2}$	$[0,0,0,0,1]$

(B.4)

$\ell = 4$:

Δ	$\mathcal{R}$
4	$[3,0,0,0,0]^*$
$\frac{7}{2}$	$[1,0,0,0,1]^*$
3	$[0,1,0,0,0]$

(B.5)

$\ell = 5$:

Δ	$\mathcal{R}$
5	$[4,0,0,0,0]^*$
$\frac{9}{2}$	$[2,0,0,0,1]^*$
4	$[0,0,1,0,0] + [1,1,0,0,0]^*$
$\frac{7}{2}$	$[1,0,0,0,1]$
3	$[0,0,0,0,0]$

(B.6)

$\ell = 6$:

Δ	$\mathcal{R}$
6	$[5,0,0,0,0]^*$
$\frac{11}{2}$	$[3,0,0,0,1]^*$
5	$[1,0,1,0,0]^* + [2,1,0,0,0]^*$
$\frac{9}{2}$	$[0,1,0,0,1] + [1,0,0,1,0] + [2,0,0,0,1]^*$
4	$[0,0,0,0,2] + [1,0,0,0,0]^* + [1,1,0,0,0]$
$\frac{7}{2}$	$[0,0,0,1,0]$

(B.7)

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$\ell = 7:$

Δ	$\mathcal{R}$
7	$[6,0,0,0,0]^*$
$\frac{13}{2}$	$[4,0,0,0,1]^*$
6	$[2,0,1,0,0]^* + [3,1,0,0,0]^*$
$\frac{11}{2}$	$[0,1,0,1,0] + [1,1,0,0,1]^* + [2,0,0,1,0]^* + [3,0,0,0,1]^*$
5	$[0,0,0,1,1] + [0,2,0,0,0] + [1,0,0,0,2]^* + [1,0,1,0,0]+$ $+[2,0,0,0,0] + [2,0,0,0,0]^* + [2,1,0,0,0]^*$
$\frac{9}{2}$	$[0,0,0,0,1] + [0,1,0,0,1] + [1,0,0,1,0]^* + [2,0,0,0,1]$
4	$[0,0,1,0,0] + [1,0,0,0,0]$

(B.8)

$\ell = 8:$

Δ	$\mathcal{R}$
8	$[7,0,0,0,0]^*$
$\frac{15}{2}$	$[5,0,0,0,1]^*$
7	$[3,0,1,0,0]^* + [4,1,0,0,0]^*$
$\frac{13}{2}$	$[1,1,0,1,0]^* + [2,1,0,0,1]^* + [3,0,0,1,0]^* + [4,0,0,0,1]^*$
6	$[0,0,0,2,0] + [0,1,1,0,0] + [1,0,0,1,1]^* + [1,1,0,0,0] + [1,2,0,0,0]^*+$ $+[2,0,0,0,2]^* + [2,0,1,0,0]^* + 2\cdot[3,0,0,0,0]^* + [3,1,0,0,0]^*$
$\frac{11}{2}$	$[0,0,1,0,1] + [0,1,0,1,0] + [1,0,0,0,1] + [1,0,0,0,1]^* + [1,1,0,0,1] + [1,1,0,0,1]^*+$ $+[2,0,0,1,0] + [2,0,0,1,0]^* + [3,0,0,0,1]^*$
5	$[0,0,0,1,1] + 2\cdot[0,1,0,0,0] + [1,0,0,0,2] + [1,0,1,0,0] + [1,0,1,0,0]^*+$ $+[2,0,0,0,0]^* + [2,1,0,0,0]$
$\frac{9}{2}$	$[0,0,0,0,1] + [0,1,0,0,1] + [1,0,0,1,0]$
4	$[1,0,0,0,0]$

(B.9)

$\ell = 9:$

Δ	$\mathcal{R}$
9	$[8,0,0,0,0]^*$
$\frac{17}{2}$	$[6,0,0,0,1]^*$
8	$[4,0,1,0,0]^* + [5,1,0,0,0]^*$
$\frac{15}{2}$	$[2,1,0,1,0]^* + [3,1,0,0,1]^* + [4,0,0,1,0]^* + [5,0,0,0,1]^*$
7	$[0,2,0,0,0] + [1,0,0,2,0]^* + [1,1,1,0,0]^* + [2,0,0,1,1]^* + [2,1,0,0,0]^*+$ $+[2,2,0,0,0]^* + [3,0,0,0,2]^* + [3,0,1,0,0]^* + 2\cdot[4,0,0,0,0]^* + [4,1,0,0,0]^*$
$\frac{13}{2}$	$[0,0,1,1,0] + [0,1,0,0,1] + [0,2,0,0,1] + [1,0,0,1,0] + [1,0,1,0,1]^* + [1,1,0,1,0]+$ $+[1,1,0,1,0]^* + 2\cdot[2,0,0,0,1]^* + 2\cdot[2,1,0,0,1]^* + 2\cdot[3,0,0,1,0]^* + [4,0,0,0,1]^*$
6	$2\cdot[0,0,1,0,0] + [0,1,0,0,2] + [0,1,1,0,0] + 2\cdot[1,0,0,1,1] + [1,0,0,1,1]^*+$ $+[1,1,0,0,0] + 2\cdot[1,1,0,0,0]^* + [1,2,0,0,0] + [2,0,0,0,2]^* + [2,0,1,0,0]+$ $+2\cdot[2,0,1,0,0]^* + [3,0,0,0,0] + [3,0,0,0,0]^* + [3,1,0,0,0]^*$
$\frac{11}{2}$	$[0,0,0,1,0] + 2\cdot[0,0,1,0,1] + 2\cdot[0,1,0,1,0] + 2\cdot[1,0,0,0,1] + [1,0,0,0,1]^*+$ $+[1,1,0,0,1] + [1,1,0,0,1]^* + [2,0,0,1,0] + [2,0,0,1,0]^* + [3,0,0,0,1]$
5	$[0,0,0,0,0] + [0,0,0,1,1] + 2\cdot[0,1,0,0,0] + [0,2,0,0,0] + [1,0,0,0,2] + [1,0,1,0,0]+$ $+[2,0,0,0,0] + [2,0,0,0,0]^*$
$\frac{9}{2}$	$[0,0,0,0,1] + [1,0,0,1,0]$

(B.10)

$\ell = 10$:

Δ	$\mathcal{R}$
10	$[9,0,0,0,0]^*$
$\frac{19}{2}$	$[7,0,0,0,1]^*$
9	$[5,0,1,0,0]^* + [6,1,0,0,0]^*$
$\frac{17}{2}$	$[3,1,0,1,0]^* + [4,1,0,0,1]^* + [5,0,0,1,0]^* + [6,0,0,0,1]^*$
8	$[1,2,0,0,0]^* + [2,0,0,2,0]^* + [2,1,1,0,0]^* + [3,0,0,1,1]^* + [3,1,0,0,0]^* +$ $+[3,2,0,0,0]^* + [4,0,0,0,2]^* + [4,0,1,0,0]^* + 2\cdot[5,0,0,0,0]^* + [5,1,0,0,0]^*$
$\frac{15}{2}$	$[0,1,0,1,0] + [0,2,0,1,0] + [1,0,1,1,0]^* + [1,1,0,0,1]^* + [1,2,0,0,1]^* +$ $+[2,0,0,1,0]^* + [2,0,1,0,1]^* + 2\cdot[2,1,0,1,0]^* + 2\cdot[3,0,0,0,1]^* + 2\cdot[3,1,0,0,1]^* +$ $+2\cdot[4,0,0,1,0]^* + [5,0,0,0,1]^*$
7	$[0,0,0,1,1] + [0,1,0,0,0] + 2\cdot[0,1,0,1,1] + [0,2,0,0,0] + [0,3,0,0,0] + [1,0,0,2,0] +$ $+[1,0,1,0,0] + 2\cdot[1,0,1,0,0]^* + [1,1,0,0,2]^* + [1,1,1,0,0] + [1,1,1,0,0]^* +$ $+3\cdot[2,0,0,1,1]^* + [2,1,0,0,0] + 3\cdot[2,1,0,0,0]^* + [2,2,0,0,0]^* + [3,0,0,0,2]^* +$ $+3\cdot[3,0,1,0,0]^* + 2\cdot[4,0,0,0,0]^* + [4,1,0,0,0]^*$
$\frac{13}{2}$	$[0,0,0,1,2] + 2\cdot[0,0,1,1,0] + 3\cdot[0,1,0,0,1] + [0,2,0,0,1] + 2\cdot[1,0,0,1,0] +$ $+[1,0,0,1,0]^* + [1,0,1,0,1] + 2\cdot[1,0,1,0,1]^* + 2\cdot[1,1,0,1,0] + 2\cdot[1,1,0,1,0]^* +$ $+2\cdot[2,0,0,0,1] + 3\cdot[2,0,0,0,1]^* + [2,1,0,0,1] + 2\cdot[2,1,0,0,1]^* + [3,0,0,1,0] +$ $+2\cdot[3,0,0,1,0]^* + [4,0,0,0,1]^*$
6	$2\cdot[0,0,0,0,2] + [0,0,0,2,0] + 2\cdot[0,0,1,0,0] + [0,1,0,0,2] + 3\cdot[0,1,1,0,0] +$ $+[1,0,0,0,0] + [1,0,0,0,0]^* + 3\cdot[1,0,0,1,1] + [1,0,0,1,1]^* + 3\cdot[1,1,0,0,0] +$ $+2\cdot[1,1,0,0,0]^* + [1,2,0,0,0]^* + [2,0,0,0,2] + [2,0,0,0,2]^* + [2,0,1,0,0] +$ $+[2,0,1,0,0]^* + [3,0,0,0,0] + 2\cdot[3,0,0,0,0]^* + [3,1,0,0,0]$
$\frac{11}{2}$	$[0,0,0,0,3] + 2\cdot[0,0,0,1,0] + [0,0,1,0,1] + 2\cdot[0,1,0,1,0] + 3\cdot[1,0,0,0,1] +$ $+[1,0,0,0,1]^* + 2\cdot[1,1,0,0,1] + [2,0,0,1,0] + [2,0,0,1,0]^*$
5	$[0,0,0,1,1] + 2\cdot[0,1,0,0,0] + [1,0,1,0,0] + [2,0,0,0,0]$
$\frac{9}{2}$	$[0,0,0,0,1]$

(B.11)

C. String partition function

In this appendix, we evaluate the string partition function

$$\mathcal{Z}(q_1, q_2) = \mathrm{Tr}\left[(-)^F\, q_1^{\ell-\bar{\ell}}\, q_2^{\nu+\bar{\nu}}\right], \tag{C.1}$$

with $q_1 = e^{2\pi i \tau_1}$, $q_2 = e^{-2\pi\tau_2}$, $\ell = L_0 - c/24 = \sum_n nN_n$, $\bar{\ell} = \bar{L}_0 - c/24 = \sum_n n\bar{N}_n$, $\nu = \sum_n N_n w_n$, $\bar{\nu} = \sum_n \bar{N}_n w_n$ the left-right moving string levels and excitation numbers corresponding to the chiral contributions to the worldsheet momentum and hamiltonian, respectively. Physical states are identified by the level matching condition $\ell = \bar{\ell}$. We start by considering the GS string in flat space ($w_n = n$):

$$\mathcal{Z}_{\rm flat}(q_1, q_2) = (\mathbf{8}_v - \mathbf{8_c})^2 \prod_{n=1}^{\infty} \frac{(1 - q_1^n q_2^n)^{\mathbf{8_s}}(1 - q_1^{-n} q_2^n)^{\mathbf{8_s}}}{(1 - q_1^n q_2^n)^{\mathbf{8_v}}(1 - q_1^{-n} q_2^n)^{\mathbf{8_v}}}, \tag{C.2}$$

with

$$\begin{aligned} (1-q^n)^{\mathbf{8_s}} &= 1 - \mathbf{8_s}\, q^n + \mathbf{8_s} \wedge \mathbf{8_s}\, q^{2n} + \cdots \\ (1-q^n)^{-\mathbf{8_v}} &= 1 + \mathbf{8_v}\, q^n + \mathbf{8}_{(\mathbf{v}} \times \mathbf{8_v})\, q^{2n} + \cdots . \end{aligned} \tag{C.3}$$

As argued in the main text, the AdS string partition function can be found from that of string theory in flat space after replacing string frequencies w_n by those of strings on plane wave at $g_{\rm YM}=0$ i.e. $w_n \to 1$ and extending the product in (C.2) to $n=0$ to account for the new zero modes:

$$q_2^n \to q_2\,, \qquad (\mathbf{8}_v - \mathbf{8_c})^2 \to \frac{(1-q_2)^{\mathbf{8_c}}}{(1-q_2)^{\mathbf{8_v}}}\,. \tag{C.4}$$

We are interested in massive string states $n \geq 1$. Primaries are found by dividing by $(\mathbf{8}_v - \mathbf{8}_s)^2$ and suppressing from bosonic and fermionic zero modes. Restricting to the chiral partition function one finds:

$$\begin{aligned}
\mathcal{Z}_{\rm SO(9)}(q_1,q_2) &\equiv \sum_{\ell,\nu} d_{\ell,\nu}\, q_1^\ell q_2^\nu = \frac{1}{(\mathbf{8}_v - \mathbf{8}_s)} \left[\prod_{n=1}^{\infty} \frac{(1-q_1^n q_2)^{\mathbf{8_s}}}{(1-q_1^n q_2)^{\mathbf{8_v}}} - 1 \right] \\
&= q_2\, q_1 + \left(q_2 + 8\, q_2^2\right)\, q_1^2 + \left(-16\, q_2^2 + q_2 + 43\, q_2^3\right)\, q_1^3 + \cdots\,.
\end{aligned} \tag{C.5}$$

This is to be compared with the third line in (2.18), e.g. at $\ell=2$ i.e. q_1^2, we find $q_2 + 8\,q_2^2 \leftrightarrow [0000]_1 + [1000]_2$. Expanding (C.5) one can easily extend this result to higher string levels. At $q_2=1$ states organized in SO(9) representations as expected. Finally the lift to SO(10) can be implemented by multiplying and dividing by $(1-q_1 q_2^2)$

$$\begin{aligned}
\mathcal{Z}_{\rm SO(10)}(q_1,q_2) &= \frac{(1-q_1 q_2^2)}{(1-q_1 q_2^2)}\, \mathcal{Z}_{\rm SO(9)}(q_1,q_2) \\
&= \frac{1}{(\mathbf{8}_v - \mathbf{8}_s)}\, \frac{(1-q_1 q_2^2)}{(1-q_1 q_2^2)} \left[\prod_{n=1}^{\infty} \frac{(1-q_1^n q_2)^{\mathbf{8_s}}}{(1-q_1^n q_2)^{\mathbf{8_v}}} - 1 \right],
\end{aligned} \tag{C.6}$$

i.e. states at level q_1^ℓ are added and subtracted at level $q_1^{\ell+1}$ e.g. at level $\ell=2$, we have the lift $q_2 + 8\,q_2^2 \to q_2 + 8\,q_2^2 + q_2^3 - q_2^3$ in agreement with the second line in (2.18) with the power ν of q_2 given as $\nu = \Delta - J$. This completes SO(10) representations as can be easily seen by expanding (C.6) and setting $q_2 \to 1$ (keeping states with negative multiplicities). The string partition function (C.6) can be used to read off multiplicities and $\mathrm{SO}(8)\times\mathrm{SO}(2)_{\Delta-J}$ charges. The assignments of conformal dimensions Δ to string states then follows from the requirement that SO(10) string representations reduce consistently to (C.5) via (2.17).

D. Free spectrum of $\mathcal{N}=4$ SYM

In this appendix we present the spectrum of long multiplets in $\mathcal{N}=4$ SYM in terms of its superconformal primaries, i.e. $\mathcal{Z}_{\rm sconf}(t,y_i)$. We have computed this spectrum up to $\Delta=10$, but for reasons of limited space, here we give the result only up to $\Delta=\frac{13}{2}$.

A long multiplet of $\mathcal{N}=4$ is described by the symbol

$$[s_1,s_2;q_1,p,q_2]^P_{L,B}\,. \tag{D.1}$$

Here, $[s_1,s_2]$ and $[q_1,p,q_2]$ indicate the Dynkin labels of SO(4) and SO(6), respectively. In addition we write the parity P, hypercharge B and length L of a state. Parity is described in [39], the hypercharge B corresponds to the leading order $\mathrm{U}(1)_B$ charge in the

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decomposition $\mathrm{SU}(2,2|4) = \mathrm{U}(1)_B \ltimes \mathrm{PSU}(2,2|4)$, and the length L is the leading order number of letters used to construct the state. Furthermore, $P = \pm$ indicates a pair with opposite parities and +conj. indicates a conjugate state $[s_2, s_1; q_2, p, q_1]^P_{L,-B}$. We group the multiplets according to their classical dimension Δ.

Δ	$\mathcal{R}$
2	$[0,0;0,0,0]^+_{2,0}$
3	$[0,0;0,1,0]^-_{3,0}$
4	$2{\cdot}[4;0,0;0,0,0]^+_{4,0} + [0,0;1,0,1]^-_{4,0} + 2{\cdot}[4;0,0;0,2,0]^+_{4,0}+$
	$+([0,2;0,0,0]^-_{3,-1} + \text{conj.}) + [1,1;0,1,0]^\pm_{3,0} + [2,2;0,0,0]^+_{2,0}$
5	$4{\cdot}[0,0;0,1,0]^-_{5,0} + 2{\cdot}([0,0;0,0,2]^+_{5,0} + \text{conj.}) + [0,0;1,1,1]^\pm_{5,0}+$
	$+\,2{\cdot}[5;0,0;0,3,0]^-_{5,0} + 2{\cdot}([0,2;0,1,0]^+_{4,-1} + \text{conj.})+$
	$+([0,2;2,0,0]^-_{4,-1} + \text{conj.}) + [1,1;0,0,0]^\pm_{4,0} + 2{\cdot}[1,1;1,0,1]^\pm_{4,0}+$
	$+[1,1;0,2,0]^\pm_{4,0} + [2,2;0,1,0]^-_{3,0}$
$\frac{11}{2}$	$2{\cdot}[0,1;1,0,0]^\pm_{5,-1/2} + 2{\cdot}[0,1;0,1,1]^\pm_{5,-1/2} + [0,1;1,2,0]^\pm_{5,-1/2}+$
	$+\,2{\cdot}[1,2;0,0,1]^\pm_{4,-1/2} + [1,2;1,1,0]^\pm_{4,-1/2} + [2,3;1,0,0]^\pm_{3,-1/2} + \text{conjugates}$
6	$2{\cdot}[0,0;0,0,0]^+_{4,0} + 2{\cdot}([0,0;0,0,0]^+_{5,-1} + \text{conj.}) + 5{\cdot}[0,0;0,0,0]^+_{6,0} + 3{\cdot}[0,0;1,0,1]^+_{6,0}+$
	$+\,6{\cdot}[0,0;1,0,1]^-_{6,0} + 9{\cdot}[0,0;0,2,0]^+_{6,0} + [0,0;0,2,0]^-_{6,0} + 3{\cdot}([0,0;0,1,2]^-_{6,0} + \text{conj.})+$
	$+\,3{\cdot}[0,0;2,0,2]^+_{6,0} + [0,0;1,2,1]^+_{6,0} + 2{\cdot}[0,0;1,2,1]^-_{6,0} + 3{\cdot}[0,0;0,4,0]^+_{6,0}+$
	$+\,2{\cdot}([0,2;0,0,0]^-_{4,0} + \text{conj.}) + 3{\cdot}([0,2;0,0,0]^-_{5,-1} + \text{conj.})+$
	$+\,4{\cdot}([0,2;1,0,1]^+_{5,-1} + \text{conj.}) + 2{\cdot}([0,2;1,0,1]^-_{5,-1} + \text{conj.})+$
	$+\,4{\cdot}([0,2;0,2,0]^-_{5,-1} + \text{conj.}) + ([0,2;2,1,0]^+_{5,-1} + \text{conj.}) + 8{\cdot}[1,1;0,1,0]^\pm_{5,0}+$
	$+\,2{\cdot}([1,1;0,0,2]^\pm_{5,0} + \text{conj.}) + 4{\cdot}[1,1;1,1,1]^\pm_{5,0} + 2{\cdot}[1,1;0,3,0]^\pm_{5,0}+$
	$+([0,4;0,0,0]^+_{3,-1} + \text{conj.}) + ([0,4;0,0,0]^+_{4,-2} + \text{conj.}) + 2{\cdot}([1,3;0,1,0]^\pm_{4,-1} + \text{conj.})+$
	$+\,5{\cdot}[2,2;0,0,0]^+_{4,0} + 2{\cdot}[2,2;0,0,0]^-_{4,0} + 2{\cdot}[2,2;1,0,1]^-_{4,0} + 4{\cdot}[2,2;0,2,0]^+_{4,0}+$
	$+[2,2;0,2,0]^-_{4,0} + ([2,4;0,0,0]^-_{3,-1} + \text{conj.}) + [3,3;0,1,0]^\pm_{3,0} + [4,4;0,0,0]^+_{2,0}$
$\frac{13}{2}$	$4{\cdot}[0,1;0,0,1]^\pm_{5,+1/2} + 6{\cdot}[0,1;0,0,1]^\pm_{6,-1/2} + 12{\cdot}[0,1;1,1,0]^\pm_{6,-1/2}+$
	$+\,5{\cdot}[0,1;1,0,2]^\pm_{6,-1/2} + [0,1;3,0,0]^\pm_{6,-1/2} + 5{\cdot}[0,1;0,2,1]^\pm_{6,-1/2}+$
	$+\,2{\cdot}[0,1;2,1,1]^\pm_{6,-1/2} + [0,1;1,3,0]^\pm_{6,-1/2} + [0,3;0,0,1]^\pm_{4,-1/2}+$
	$+[0,3;0,0,1]^\pm_{5,-3/2} + 2{\cdot}[0,3;1,1,0]^\pm_{5,-3/2} + 10{\cdot}[1,2;1,0,0]^\pm_{5,-1/2}+$
	$+\,8{\cdot}[1,2;0,1,1]^\pm_{5,-1/2} + 3{\cdot}[1,2;2,0,1]^\pm_{5,-1/2} + 2{\cdot}[1,2;1,2,0]^\pm_{5,-1/2}+$
	$+[1,4;1,0,0]^\pm_{4,-3/2} + 3{\cdot}[2,3;0,0,1]^\pm_{4,-1/2} + 2{\cdot}[2,3;1,1,0]^\pm_{4,-1/2} + \text{conjugates}$

(D.2)

E. Tools for representations

In this appendix we present two algorithms to construct and deconstruct multiplets of $\mathrm{SO}(2n)$.

E.1 Construction

The character polynomial $\chi_{\mathbf{w}}$ associated with the highest weight state $\mathbf{y}^{\mathbf{w}}$ may be directly obtained from Weyl's character formula. For $\mathrm{SO}(2n)$, this takes the form

$$\chi_{\mathbf{w}} = \frac{\sum_W (-)^{|W|}\, W(\mathbf{y}^{\mathbf{w}+\rho})}{\sum_W (-)^{|W|}\, W(\mathbf{y}^{\rho})}\,, \tag{E.1}$$

where $\rho = (n{-}1, n{-}2, \ldots, 0)$ denotes one half the sum of the positive roots. The sums are running over the Weyl group of $\mathrm{SO}(2n)$, which is generated by the group S_n of permutations of the y_n, together with the element $\{y_1 \to y_1^{-1}\,,\ y_2 \to y_2^{-1}\}$.

Another way to construct a multiplet is as follows. We first decompose a highest weight multiplet of SO$(2n)$ into multiplets of SO$(2n-2) \times$ SO(2). This can be used to recursively construct all the weights in a highest weight multiplet of SO$(2n)$. The recursion formula is

$$\chi_{(w_1,\ldots,w_n)} = \sum_{w_i'} \chi_{(w_1',\ldots,w_{n-1}')} y_n^{c_n} \prod_{i=1}^{n-1} \chi_{(c_i)} \,, \tag{E.2}$$

with $\chi_{(c_i)}$ given by

$$\chi_{(c_i)} = \frac{(y_n^{-c_i} - y_n^{c_i+2})}{(1-y_n^2)} \,. \tag{E.3}$$

The coefficients c_i determining the range of SO(2) charges are given by

$$\begin{aligned} c_1 &= w_1 - \max(w_1', w_2) \geq 0 \,, \\ &\vdots \\ c_i &= \min(w_i, w_{i-1}') - \max(w_i', w_{i+1}) \geq 0 \,, \\ &\vdots \\ c_{n-1} &= \min(w_{n-1}, w_{n-2}') - \max(|w_{n-1}'|, |w_n|) \geq 0 \,, \\ c_n &= \operatorname{sign} w_n \, \operatorname{sign} w_{n-1}' \, \min(|w_n|, |w_{n-1}'|) \,, \end{aligned} \tag{E.4}$$

and the sum runs over those values of w_i' for which the coefficients $c_1, \ldots, c_{n-1}$ are non-negative.

E.2 Deconstruction

The highest weight $\mathbf{w}$ of a irreducible multiplet $\chi_{\mathbf{w}}(y_i)$ of SO$(2n)$, $i = 1, \ldots, n$ can be obtained by a simple algorithm (alternating dominant) that yields

$$\chi_{\mathbf{w}} \mapsto (\chi_{\mathbf{w}})_{\mathrm{HW}} = \mathbf{y}^{\mathbf{w}} \,. \tag{E.5}$$

We start by defining the fundamental Weyl chamber for a state $\mathbf{y}^{(w_1,\ldots,w_n)}$ of SO$(2n)$ by the condition

$$w_1 \geq w_2 \geq \ldots \geq w_{n-1} \geq |w_n| \,. \tag{E.6}$$

The algorithm moves all charges to the fundamental chamber by Weyl reflections. These interchange two SO(2) charges w_k, w_l in the following way

$$\begin{aligned} y_k^{w_k} y_l^{w_l} &\mapsto -y_k^{w_l+k-l} y_l^{w_k+l-k} \qquad \text{for} \quad k \neq l \,, \\ y_k^{w_k} y_n^{w_n} &\mapsto -y_k^{-w_n+k-n} y_n^{-w_k+k-n} \,. \end{aligned} \tag{E.7}$$

Charges at the boundaries of chambers, $w_l = w_k - k + l$ or $w_n = -w_k + k - n$, are dropped, $y_k^{w_k} y_l^{w_k-k+l} \mapsto 0, y_k^{w_k} y_n^{-w_k+k+n} \mapsto 0$.

When all the charges have been reflected to the fundamental chamber, only one component, $\mathbf{y}^{\mathbf{w}}$, the highest weight state, should remain. All the other components will have canceled among themselves. As this algorithm is linear, it can equally well be used to decompose any reducible representation of SO$(2n)$ into the highest weights of the irreducible components

$$\left(\sum_k \chi_{\mathbf{w}_k}\right)_{\mathrm{HW}} = \sum_k \mathbf{y}^{\mathbf{w}_k} . \tag{E.8}$$

Especially, this algorithm can be applied to decompose the $\mathcal{N} = 4$ partition function eq. (3.3) into irreducible representations of SO$(6) \times$ SO(4)

$$\mathcal{Z}_{\mathrm{SYM}}(t, y_i) \mapsto \mathcal{Z}_{\mathrm{SYM}}(t, y_i)_{\mathrm{HW}} . \tag{E.9}$$

To this end it is particularly useful the following property (Klimyk's formula):

$$(\chi\chi')_{\mathrm{HW}} = (\chi_{\mathrm{HW}}\chi')_{\mathrm{HW}} , \tag{E.10}$$

that allow us to determine the tensor product of two multiplets, without computing the full product of their character polynomials. In other words, to determine the tensor product, we only need to compute the product of the highest weights of one multiplet with all the weights of the other multiplet. In our notation:

$$(\chi_{\mathbf{w}}\chi_{\mathbf{w}'})_{\mathrm{HW}} = (\mathbf{y}^{\mathbf{w}}\chi_{\mathbf{w}'})_{\mathrm{HW}} . \tag{E.11}$$

This can be used to significantly simplify expression eq. (3.4) when we ask only for highest weights with respect to SO$(6) \times$ SO(4). According to eq. (E.10) and eq. (E.5) only the highest weight $\mathbf{y}^{\mathbf{w}}$ enter in this reduction and there is no need to construct the full multiplets of SO$(4) \times$ SO(6). The result can then be read as:

$$\chi^{\Delta}_{\mathbf{w}}(t, y_i)_{\mathrm{HW}6\times 4} = \left(\mathcal{T}_{\mathrm{sconf}}(t, y_i)\, t^{\Delta}\, \chi_{\mathbf{w}}\right)_{\mathrm{HW}6\times 4} = \left(\mathcal{T}_{\mathrm{sconf}}(t, y_i)\, t^{\Delta}\, \mathbf{y}^{\mathbf{w}}\right)_{\mathrm{HW}6\times 4} . \tag{E.12}$$

In a similar way one proceeds for BPS or semishort multiplets, with final result again eq. (E.12) but with the product over s restricted to a subset of the supercharges $\mathbf{w}_s$. For example the highest weight content of a $\frac{1}{2}$-BPS multiplet can be written as:

$$\chi^{n}_{(n00;00)}(t, y_i)_{\mathrm{HW}6\times 4} = \left(t^n y_1^n \frac{\prod_{s=1}^{8}(1 - t^{1/2}\mathbf{y}^{\mathbf{w}_s})}{\prod_{v=1}^{4}(1 - t\,\mathbf{y}^{\mathbf{w}_v})}\right)_{\mathrm{HW}6\times 4} . \tag{E.13}$$

The product over supercharges is restricted to those $\mathbf{w}_s$ in eq. (3.6) with $w_1 = -\frac{1}{2}$ i.e. the unbroken supercharges. Notice that expressions inside brackets $()_{\mathrm{HW}}$ typically involve incomplete multiplets of SO$(4) \times$ SO(6). It is somewhat remarkable that, after removing negative weights, the correct highest weights are produced by this expression.

Finally the algorithm provides an unambiguous procedure to flip between irreps of SO$(6) \times$ SO(4) and SO(10) when the SO$(6) \times$ SO(4) representation admits such a lift. For

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that we identify states of SO(10) and SO(4) × SO(6) by $\mathbf{y}_{(w_1,w_2,w_3,w_4,w_5)} \equiv \mathbf{y}_{(w_1,w_2,w_3)} \times \mathbf{y}_{(w_4,w_5)}$. To decompose a multiplet $\chi_{6\times 4}$ of SO(6) × SO(4) into highest weight states of SO(10) or vice versa we use

$$(\chi_{6\times 4})_{\mathrm{HW10}} \qquad \text{or} \qquad (\chi_{10})_{\mathrm{HW6\times 4}} \,. \tag{E.14}$$

This can be applied to $\mathcal{Z}_{\mathrm{SO}(10,2)}(t, y_i)$ in (3.12) to read the SO(10) content of SYM primaries up to KK recurrences $\mathcal{Z}_{\mathrm{SO}(10,2)}(t, y_i)_{\mathrm{HW10}}$.

F. Cyclic words

In this appendix we present a complete set of representatives for cyclic words. We construct the weights of all single trace operators in $\mathcal{N} = 4$ SYM. The module χ_L of all such cyclic words of length L is constructed as

$$\chi_L = \sum_{W_i \in \chi_{\mathrm{F}}} \mathrm{Tr}(W_1 \ldots W_L) \,. \tag{F.1}$$

Each letter W_i is summed over all fundamental fields in

$$\chi_{\mathrm{F}} = \{\partial^s \phi, \partial^s \lambda, \partial^s \bar{\lambda}, \partial^s F\} \,. \tag{F.2}$$

Naively, we overcount due to cyclicity of traces. Therefore we define Tr as to select exactly one representative of each cyclic class. This can be done as follows

$$\mathrm{Tr}(W_1 \ldots W_L) = 0 \quad \text{unless} \quad (W_1 \ldots W_L) \leq (W_{i+1} \ldots W_L W_1 \ldots W_i) \text{ for all } i \,, \tag{F.3}$$

where we define some ordering on the set of words. Furthermore we need to determine whether the operator can exist at all due to statistics. If we can write $\mathrm{Tr}(W_1 \ldots W_L)$ as $\mathrm{Tr}[(W_1 \ldots W_{L/2})^2]$. Then

$$\mathrm{Tr}\left[(W_1 \ldots W_{L/2})^2\right] = 0 \,, \qquad \text{if } (W_1 \ldots W_{L/2}) \text{ is fermionic} \,. \tag{F.4}$$

Exactly in this case we can commute one block around the trace and pick up a sign.

Finally, we would like to determine the parity P of an operator. For an operator $\mathrm{Tr}(W_1 \ldots W_L)$ to have a definite parity we need to have

$$(W_1 \ldots W_L) = (W_i W_{i-1} \ldots W_2 W_1 W_L W_{L-1} \ldots W_{i+1}) \tag{F.5}$$

If this holds, parity is given by

$$P = (-1)^{[n/2]+[n'/2]+L} \,, \tag{F.6}$$

where n and n' are the numbers of fermions in $W_1 \ldots W_i$ and $W_{i+1} \ldots W_L$, respectively. This number is obtained by reversing the order of $n+n'$ fermions and commuting n fermions across n'. Furthermore, each field has negative intrinsic parity, thus the factor of $(-1)^L$. If eq. (F.5) does not hold, only the two combinations of states

$$\mathrm{Tr}(W_1 \ldots W_L) \pm \mathrm{Tr}(W_L \ldots W_1) \tag{F.7}$$

have define and opposite parity. For counting purposes, we should therefore assign some parity to $\mathrm{Tr}(W_1 \dots W_L)$ and the opposite one to the reversed operator $\mathrm{Tr}(W_L \dots W_1)$. We assign positive parity, $P = +$, to $\mathrm{Tr}(W_1 \dots W_L)$ if

$$(W_1 \dots W_L) < (W_i W_{i-1} \dots W_2 W_1 W_L W_{L-1} \dots W_{i+1}) \quad \text{for all } i\,, \tag{F.8}$$

and negative otherwise. In this way each pair of operators receives both parities.

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